Problem Set 8 due May 6, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

For any angles α and β, consider the complex numbers:

$$z = \cos\alpha + i\sin\alpha \qquad \text{and} \qquad w = \cos\beta + i\sin\beta$$

(1) Convert z and w to polar form, and compute zw in polar form. Draw z, w and zw on a picture of the complex plane, indicating their absolute value and arguments. *(10 points)*

(2) Convert zw into its Cartesian form, and use this fact to obtain formulas for:

$$\cos(\alpha + \beta) = \ldots$$
$$\sin(\alpha + \beta) = \ldots$$

in terms of $\cos\alpha$, $\sin\alpha$, $\cos\beta$, $\sin\beta$. *(10 points)*

Problem 2:

Fix numbers a and b. Write the symmetric matrix:

$$S = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}$$

explicitly as $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal. Explain all of your steps (*Hint: the characteristic polynomial of a* 4×4 *matrix is a degree 4 polynomial, and therefore difficult in general to solve; however, in the case at hand, it will be easy to find its roots*). *(20 points)*

Problem 3:

Consider the 3×3 symmetric matrix S such that:

$$\begin{bmatrix} x & y & z \end{bmatrix} S \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (3x - 2y + z)^2$$

for any x, y, z.

(1) Without doing any computations on S, explain why S cannot have full rank. *(5 points)*

(2) Write S out explicitly. *(5 points)*

(3) Compute the eigenvalues and eigenvectors of S. *(10 points)*

(4) Does your answer in part (3) agree with part (1)? Explain why. Is S positive definite, positive semi-definite, or neither? *(5 points)*

Problem 4:

Represent the US flag as a 13×25 matrix A, where each entry represents a color as follows: the entry 1 represents red, the entry 0 represents white, and the entry -1 represents blue. Then write this matrix A as a sum of rank 1 matrices.

Note on vexillology: you may ignore the stars, so just assume that the top left corner is a full-blue 7×10 submatrix of A. The height of all the stripes is one row. *(15 points)*

Problem 5:

All matrices in this problem are 2×2. A lower/upper triangular matrix with 1's on the diagonal has one degree of freedom (the bottom-left/top-right entry); a diagonal matrix has two degrees of freedom (the diagonal entries). Hence the LDU factorization has $1 + 2 + 1$ degrees of freedom, which is precisely the number of degrees of freedom in choosing a 2×2 matrix.

(1) How many degrees of freedom does an orthogonal 2×2 matrix Q have? Explain. *(5 points)*

(2) What is the total number of degrees of freedom of the QR factorization? What about the total number of degrees of freedom of the SVD $U\Sigma V^T$? Explain. *(6 points)*

(3) What is the total number of degrees of freedom of $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal? Still in the 2×2 case. *(4 points)*

(4) Why didn't you get 4 in part (c)?

Hint: it's because matrices $Q\Lambda Q^T$ are special, i.e. they are ______________ *(5 points)*